

RESONANCE VARIETIES VIA BLOWUPS OF \mathbb{P}^2 AND SCROLLS

HAL SCHENCK

ABSTRACT. Conjectures of Suciu [36] relate the fundamental group of an arrangement complement $M = \mathbb{C}^n \setminus \mathcal{A}$ to the first resonance variety of $H^*(M, \mathbb{Z})$. We describe a connection between the first resonance variety and the Orlik-Terao algebra $C(\mathcal{A})$ of the arrangement. In particular, we show that non-local components of $R^1(\mathcal{A})$ give rise to determinantal syzygies of $C(\mathcal{A})$. As a result, $\text{Proj}(C(\mathcal{A}))$ lies on a scroll, placing geometric constraints on $R^1(\mathcal{A})$. The key observation is that $C(\mathcal{A})$ is the homogeneous coordinate ring associated to a nef but not ample divisor on the blowup of \mathbb{P}^2 at the singular points of \mathcal{A} .

1. INTRODUCTION

The fundamental group of the complement M of an arrangement of hyperplanes $\mathcal{A} = \bigcup_{i=1}^d H_i \subseteq \mathbb{C}^n$ is a much studied object. The Lefschetz-type theorem of Hamm-Le [13] implies that taking a generic two dimensional slice of M yields an isomorphism at the level of fundamental groups, so to study $\pi_1(M)$ we may assume $\mathcal{A} \subseteq \mathbb{P}^2$. Even with this simplifying assumption the situation is nontrivial: in [17] Hirzebruch writes “The topology of the complement of an arrangement of lines in \mathbb{P}^2 is very interesting, the investigation of the fundamental group very difficult”.

Presentations for $\pi_1(M)$ are given by Randell [28], Salvetti [29], Arvola [2], and Cohen-Suciu [3]. Perhaps the most compact of these is the braid monodromy presentation of [3], but even this is quite complicated. Somewhat coarser invariants of $\pi_1(M)$ are the LCS ranks and Chen ranks. For a finitely generated group G , let $G = G_1$ and define a sequence of normal subgroups inductively by $G_k = [G_{k-1}, G]$. This yields an associated Lie algebra

$$\text{gr}(G) \otimes \mathbb{Q} := \bigoplus_{k=1}^{\infty} G_k / G_{k+1} \otimes \mathbb{Q},$$

with Lie bracket induced by the commutator. The k -th LCS rank $\phi_k = \phi_k(G)$ is the rank of the k -th quotient. The Chen ranks of a group are the LCS ranks of the maximal metabelian quotient $G/[[G, G], [G, G]]$. Work of Papadima and Suciu [26] shows that the Chen ranks of $\pi_1(M)$ are combinatorially determined; but save for some special classes of arrangements, there are no explicit formulas for either the Chen or LCS ranks. However, there are a beautiful pair of conjectures due to Suciu [36], giving formulas for the LCS and Chen ranks in terms of the first resonance variety $R^1(\mathcal{A})$. The variety $R^1(\mathcal{A})$ is the tangent cone at the origin to the characteristic variety; the study of $R^1(\mathcal{A})$ was pioneered by Falk in [10].

2000 *Mathematics Subject Classification.* 13D02, 52C35, 14J26, 14C20.

Key words and phrases. Rational surface, Line arrangement, syzygy.

Schenck supported by NSF 07-07667, NSA 904-03-1-0006.

In the next section, we review the main subjects of investigation: the Orlik-Solomon algebra $A = H^*(M, \mathbb{Z})$, the Orlik-Terao algebra $C(\mathcal{A})$, the first resonance variety $R^1(\mathcal{A})$, and blowups of \mathbb{P}^2 using certain divisors. Our main result is a description of $C(\mathcal{A})$ as the homogeneous coordinate ring of the blowup X of \mathbb{P}^2 at the singular points of \mathcal{A} , via a specific (nef but not ample) divisor $D_{\mathcal{A}}$. This allows us to give a geometric interpretation of $R^1(\mathcal{A})$ in terms of certain determinantal syzygies; we prove that if \mathcal{A} supports a net, then $\text{Proj}(C(\mathcal{A}))$ lies on a scroll.

2. BACKGROUND

In [23], Orlik and Solomon gave a presentation for the cohomology ring of the complement M of a set of hyperplanes $\mathcal{A} \subseteq \mathbb{C}^n$. A consequence of their work is that the Betti numbers of M are determined by the intersection lattice $L(\mathcal{A})$. This lattice is ranked by codimension: $x \in L_i(\mathcal{A})$ corresponds to a linear space of codimension i which is an intersection of hyperplanes of \mathcal{A} . The lattice element $\hat{0}$ corresponds to \mathbb{C}^n , and $y \prec x \leftrightarrow x \subsetneq y$. We work with \mathcal{A} central, so \mathcal{A} defines an arrangement in both \mathbb{C}^n and \mathbb{P}^{n-1} . We will depict \mathcal{A} projectively, as below:

Example 2.1. The reflecting hyperplanes of the Weyl group of $SL(4)$ are the six hyperplanes in \mathbb{C}^4 defined by $V(x_i - x_j)$, $1 \leq i < j \leq 4$. Projecting along the common subspace (t, t, t, t) yields the *braid arrangement* of six planes containing the origin in \mathbb{C}^3 or six lines in \mathbb{P}^2 :

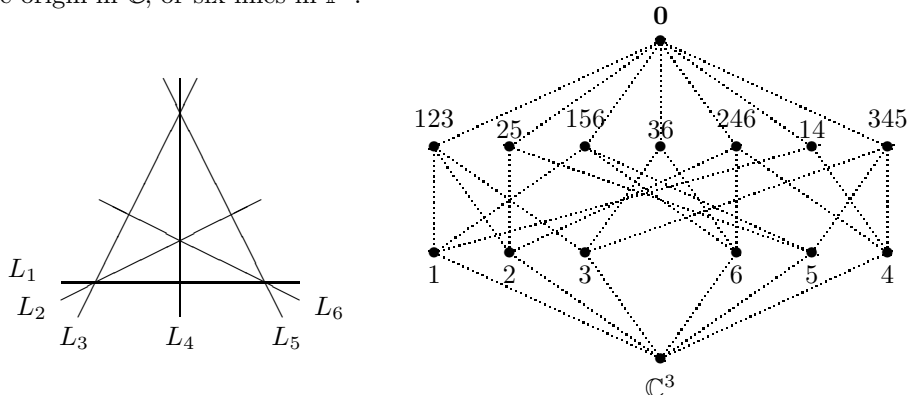


FIGURE 1. The braid arrangement A_3 and its intersection lattice in \mathbb{C}^3

Definition 2.2. The Möbius function $\mu : L(\mathcal{A}) \rightarrow \mathbb{Z}$ is defined by

$$\begin{aligned} \mu(\hat{0}) &= 1 \\ \mu(t) &= -\sum_{s \prec t} \mu(s), \text{ if } \hat{0} \prec t \end{aligned}$$

As noted, the Poincaré polynomial of M is determined by $L(\mathcal{A})$:

$$P(M, t) = \sum_{x \in L(\mathcal{A})} \mu(x) \cdot (-t)^{\text{rank}(x)}.$$

In Example 2.1, $P(M, t) = 1 + 6t + 11t^2 + 6t^3$. For a central arrangement in \mathbb{C}^n , $M \simeq \mathbb{C}^* \times (\mathbb{P}^{n-1} \setminus \mathcal{A})$, so by Künneth $P(M, t) = (1 + t)P(\mathbb{P}^{n-1} \setminus \mathcal{A}, t)$. For $n = 3$, $b_2(M) = \sum_{p \in L_2(\mathcal{A})} \mu(p)$, where $\mu(p)$ is one less than the number of lines through p .

2.1. Orlik-Solomon algebra and $R^1(\mathcal{A})$. The Orlik and Solomon presentation for the cohomology ring of $M = \mathbb{C}^n \setminus \mathcal{A}$ is as follows:

Definition 2.3. $A = H^*(M, \mathbb{Z})$ is the quotient of the exterior algebra $E = \bigwedge(\mathbb{Z}^d)$ on generators e_1, \dots, e_d in degree 1 by the ideal generated by all elements of the form $\partial e_{i_1 \dots i_r} := \sum_q (-1)^{q-1} e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r}$, for which $\text{codim } H_{i_1} \cap \cdots \cap H_{i_r} < r$.

Since A is a quotient of an exterior algebra, multiplication by an element $a \in A^1$ gives a degree one differential on A , yielding a cochain complex (A, a) :

$$(A, a): \quad 0 \longrightarrow A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots \xrightarrow{a} A^\ell \longrightarrow 0.$$

The complex (A, a) is exact as long as $\sum_{i=1}^n a_i \neq 0$; the *first resonance variety* $R^1(\mathcal{A})$ consists of points $a = \sum_{i=1}^n a_i e_i \leftrightarrow (a_1 : \cdots : a_n)$ in $\mathbb{P}(A^1) \cong \mathbb{P}^{d-1}$ for which $H^1(A, a) \neq 0$. Falk initiated the study of $R^1(\mathcal{A})$ in [10]; among his main innovations was the concept of a *neighborly partition*: a partition Π of \mathcal{A} is neighborly if, for any rank two flat $Y \in L_2(\mathcal{A})$ and any block π of Π ,

$$\mu(Y) \leq |Y \cap \pi| \implies Y \subseteq \pi,$$

Falk showed that all components of $R^1(\mathcal{A})$ arise from such partitions, and conjectured that $R^1(\mathcal{A})$ was a subspace arrangement. This was proved, essentially simultaneously, by Cohen-Suciu [3] and Libgober-Yuzvinsky [21]; we will return to this in §4.

2.2. The Orlik-Terao algebra. In [25], Orlik and Terao introduced a commutative analog of the Orlik-Solomon algebra in order to answer a question of Aomoto.

Definition 2.4. Let $\mathcal{A} = \cup_{i=1}^d V(\alpha_i) \subseteq \mathbb{P}^n$, and put $R = \mathbb{C}[y_1, \dots, y_d]$. For each linear dependency $\Lambda = \sum_{j=1}^k c_{i_j} \alpha_{i_j} = 0$, define $f_\Lambda = \sum_{j=1}^k c_{i_j} (y_{i_1} \cdots \hat{y}_{i_j} \cdots y_{i_k})$, and let I be the ideal generated by the f_Λ . The Orlik-Terao algebra $C(\mathcal{A})$ is the quotient of $\mathbb{C}[y_1, \dots, y_d]$ by I , and the Artinian Orlik-Terao algebra (the main object studied in [25]) is $C(\mathcal{A})/\langle y_1^2, \dots, y_d^2 \rangle$.

Example 2.5. Suppose $\mathcal{A} \subseteq \mathbb{P}^2$ is defined by the vanishing of $\alpha_1 = x_1, \alpha_2 = x_2, \alpha_3 = x_3, \alpha_4 = x_1 + x_2 + x_3$. The only relation is $\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = 0$, so

$$C(\mathcal{A}) = \mathbb{C}[y_1, \dots, y_4] / \langle y_2 y_3 y_4 + y_1 y_3 y_4 + y_1 y_2 y_4 - y_1 y_2 y_3 \rangle.$$

The homogeneous polynomial $y_2 y_3 y_4 + y_1 y_3 y_4 + y_1 y_2 y_4 - y_1 y_2 y_3$ is irreducible, hence defines a cubic surface in \mathbb{P}^3 , and a computation shows that the surface has four singular points. A classical result in algebraic geometry is that the linear system of four cubics through six general points in \mathbb{P}^2 defines a map from the blowup of \mathbb{P}^2 at those points to \mathbb{P}^3 whose image is a smooth cubic surface. As the points move into special position the surface acquires singularities, as in this example.

In [34], properties of the Orlik-Terao algebra were studied in relation to 2-formality. An arrangement is 2-formal if any dependency among the linear forms defining the the arrangement can be obtained as a linear combination of dependencies which involve only three of the forms. Among the classes of 2-formal arrangements are $K(\pi, 1)$ arrangements and free arrangements. However, an example of Yuzvinsky [41] shows that 2-formality is not determined by the intersection lattice $L(\mathcal{A})$. The main result of [34] is that 2-formality is determined by the quadratic component of the Orlik-Terao ideal; the key is a computation on the tangent space of $V(I_2) \cap (\mathbb{C}^*)^{d-1}$.

2.3. Blowups of \mathbb{P}^2 . Fix points $p_1, \dots, p_n \in \mathbb{P}^2$, and let

$$(1) \quad X \xrightarrow{\pi} \mathbb{P}^2$$

be the blow up of \mathbb{P}^2 at these points. Then $\text{Pic}(X)$ is generated by the exceptional curves E_i over the points p_i , and the proper transform E_0 of a line in \mathbb{P}^2 . A classical geometric problem asks for a relationship between numerical properties of a divisor $D_m = mE_0 - \sum a_i E_i$ on X , and the geometry of

$$X \xrightarrow{\phi} \mathbb{P}(H^0(D_m)^\vee).$$

First, some basics. Let m and a_i be non-negative, let I_{p_i} denote the ideal of a point p_i , and define

$$(2) \quad J = \bigcap_{i=1}^n I_{p_i}^{a_i} \subseteq \mathbb{C}[x, y, z] = S.$$

Then $H^0(D_m)$ is isomorphic to the m^{th} graded piece J_m of J (see [14]). In [5], Davis and Geramita show that if $\gamma(J)$ denotes the smallest degree t such that J_t defines J scheme theoretically, then D_m is very ample if $m > \gamma(J)$, and if $m = \gamma(J)$, then D_m is very ample iff J does not contain m collinear points, counted with multiplicity. Note that $\gamma(J) \leq \text{reg}(J)$. Now suppose that $\mathcal{A} = \cup_{i=1}^d L_i \subseteq \mathbb{P}^2$, and fix defining linear forms α_i so that $L_i = V(\alpha_i)$. Let X denote the blowup of \mathbb{P}^2 at $\text{Sing}(\mathcal{A}) = L_2(\mathcal{A})$. The central object of our investigations is the divisor

$$(3) \quad D_{\mathcal{A}} = (d-1)E_0 - \sum_{p_i \in L_2(\mathcal{A})} \mu(p_i)E_i.$$

2.4. Main results. For an arrangement $\mathcal{A} \subseteq \mathbb{P}^2$, let

$$(4) \quad X \xrightarrow{\phi_{\mathcal{A}}} \mathbb{P}(H^0(D_{\mathcal{A}})^\vee).$$

We show that $C(\mathcal{A})$ is the homogeneous coordinate ring of $\phi_{\mathcal{A}}(X)$, and that $\phi_{\mathcal{A}}$ is an isomorphism on $\pi^*(\mathbb{P}^2 \setminus \mathcal{A})$, contracts the lines of \mathcal{A} to points, and blows up the singularities of \mathcal{A} . Combining results of Proudfoot-Speyer [27] and Terao [38], we bound the Castelnuovo-Mumford regularity of $C(\mathcal{A})$. Finally, we interpret the resonance varieties studied in [3], [10], [12], [21], [32], [42] in terms of linear subsystems of $D_{\mathcal{A}}$, and connect these jump loci to linear syzygies on $C(\mathcal{A})$.

3. CONNECTING $H^0(D_{\mathcal{A}})$ TO THE ORLIK-TERA0 ALGEBRA

Let $\alpha = \prod_{i=1}^d \alpha_i$ and define a map $R = \mathbb{C}[y_1, \dots, y_d] \longrightarrow \mathbb{C}[1/\alpha_1, \dots, 1/\alpha_d] = T$. The kernel of this map is the OT ideal (see [34]), so $C(\mathcal{A}) \simeq T$. In [38], Terao proved that the Hilbert series for T is given by

$$(5) \quad HS(T, t) = P\left(\mathcal{A}, \frac{t}{1-t}\right).$$

In this section, we show that for $n = 2$, $C(\mathcal{A})$ is the homogeneous coordinate ring of the image of $X \xrightarrow{\phi_{\mathcal{A}}} \mathbb{P}(H^0(D_{\mathcal{A}})^\vee)$, with X as in Equation 1. For brevity, let $l_i = \alpha/\alpha_i$.

Lemma 3.1. *The ideal $L = \langle l_1, \dots, l_d \rangle$ defines*

$$\bigcap_{p_i \in L_2(\mathcal{A})} I_{p_i}^{\mu(p_i)} \text{ scheme-theoretically.}$$

Proof. Localize at I_p , where $p \in L_2(\mathcal{A})$. Then in S_{I_p} , α_i is a unit if $p \notin V(\alpha_i)$. Without loss of generality, suppose forms $\alpha_1, \dots, \alpha_m$ vanish on p , and the remaining forms do not. Thus,

$$L_{I_p} = \langle \alpha_2 \cdots \alpha_m, \alpha_1 \cdot \alpha_3 \cdots \alpha_m, \dots, \alpha_1 \cdots \alpha_{m-1} \rangle.$$

Now note that $I_p^{\mu(p)}$ has $\mu(p) + 1$ generators of degree $\mu(p)$. Since $\mu(p) = m - 1$ and the forms in L_{I_p} are linearly independent, equality follows. \square

Lemma 3.2. *The minimal free resolution of S/L is*

$$0 \longrightarrow S(-d)^{d-1} \xrightarrow{\psi} S(-d+1)^d \xrightarrow{[l_1, \dots, l_d]} S \longrightarrow S/L \longrightarrow 0, \text{ where}$$

$$\psi = \begin{bmatrix} \alpha_1 & 0 & \cdots & \cdots & 0 \\ -\alpha_2 & \alpha_2 & 0 & \cdots & \vdots \\ 0 & -\alpha_3 & \ddots & \ddots & \vdots \\ \vdots & 0 & & \ddots & 0 \\ \vdots & \vdots & \ddots & & \alpha_{d-1} \\ 0 & \cdots & \cdots & 0 & -\alpha_d \end{bmatrix}$$

Proof. The columns of ψ are syzygies on L . Since the maximal minors of ψ generate L , the result follows from the Hilbert-Burch theorem and Lemma 3.1. \square

Theorem 3.3. $H^0(D_{\mathcal{A}}) \simeq \text{Span}_{\mathbb{C}}\{l_1, \dots, l_d\}$ and $H^1(D_{\mathcal{A}}) = 0 = H^2(D_{\mathcal{A}})$.

Proof. The remark following Equation 2 shows that $H^0(D_{\mathcal{A}}) \simeq J_{d-1}$. Since $K = -3E_0 + \sum E_i$, by Serre duality

$$H^2(D_{\mathcal{A}}) \simeq H^0((-d-2)E_0 + \sum_{p_i \in L_2(\mathcal{A})} (\mu(p_i) + 1)E_i),$$

which is clearly zero. Using that X is rational, it follows from Riemann-Roch that

$$h^0(D_{\mathcal{A}}) - h^1(D_{\mathcal{A}}) = \frac{D_{\mathcal{A}}^2 - D_{\mathcal{A}} \cdot K}{2} + 1.$$

The intersection pairing on X is given by $E_i^2 = 1$ if $i = 0$, and -1 if $i \neq 0$, and

$$E_i \cdot E_j = 0 \text{ if } i \neq j.$$

Thus,

$$(6) \quad \begin{aligned} D_{\mathcal{A}}^2 &= (d-1)^2 - \sum_{p \in L_2(\mathcal{A})} \mu(p)^2 \\ -D_{\mathcal{A}}K &= 3(d-1) - \sum_{p \in L_2(\mathcal{A})} \mu(p), \end{aligned}$$

yielding

$$(7) \quad \begin{aligned} h^0(D_{\mathcal{A}}) - h^1(D_{\mathcal{A}}) &= \frac{(d-1)^2 - \sum \mu(p)^2 + 3(d-1) - \sum \mu(p)}{2} + 1 \\ &= \binom{d+1}{2} - \sum_{p \in L_2(\mathcal{A})} \binom{\mu(p)+1}{2}. \end{aligned}$$

Double counting the edges between $L_1(\mathcal{A})$ and $L_2(\mathcal{A})$ yields

$$\binom{d}{2} = \sum_{p \in L_2(\mathcal{A})} \binom{\mu(p) + 1}{2},$$

hence $h^0(D_{\mathcal{A}}) - h^1(D_{\mathcal{A}}) = d$. From Lemmas 3.1 and 3.2 the Hilbert function satisfies

$$d = HF(\langle l_1, \dots, l_d \rangle, d-1) = HF\left(\bigcap_{p_i \in L_2(\mathcal{A})} I_{p_i}^{\mu(p_i)}, d-1\right).$$

The observation after Equation 2 now implies that $h^0(D_{\mathcal{A}}) = d$. \square

It follows from Theorem 3.3 that $C(\mathcal{A})$ is the coordinate ring of $\phi_{\mathcal{A}}(X)$. Note also that by Lemmas 3.1 and 3.2, the constant $\gamma(L) = d-1$, so

$$dE_0 - \sum_{p_i \in L_2(\mathcal{A})} \mu(p_i)E_i$$

is very ample, and gives a De Concini-Procesi wonderful model [6]: a compactification \overline{M} of M such that $\overline{M} \setminus M$ is a normal crossing divisor. However, since every line of \mathcal{A} contains exactly $d-1$ points counted with multiplicity, the divisor $D_{\mathcal{A}}$ is not very ample. The description of $C(\mathcal{A})$ makes it obvious that $V(I)$ is an irreducible, nondegenerate rational variety, and by [34] $V(I) \setminus V(y_1 \cdots y_d)$ is smooth. Here is a more explicit description of the map:

Theorem 3.4. *The map $\phi_{\mathcal{A}}$*

- (1) *is an isomorphism on $\pi^*(\mathbb{P}^2 \setminus \mathcal{A})$.*
- (2) *contracts the lines of \mathcal{A} to points on X .*
- (3) *takes E_p to a rational normal curve of degree $\mu(p)$.*

Proof. For the first part, without loss of generality suppose that $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = xyz$ and write $L = \prod_{i=4}^d \alpha_i$. Then

$$\phi_{\mathcal{A}} = [yzL, xzL, xyL, \dots].$$

Thus, the first three entries of $\phi_{\mathcal{A}}$ define the Cremona transformation, which is an isomorphism from $\mathbb{P}^2 \setminus V(xyz)$ to itself. Since $\mathbb{P}^2 \setminus \mathcal{A}$ is contained in $\mathbb{P}^2 \setminus V(xyz)$, (1) follows. For (2), suppose p is a point of $V(\alpha_i)$. Since α_i divides l_j for all $j \neq i$, this means $l_j(p) = 0$ if $j \neq i$. Hence $\phi_{\mathcal{A}}(V(\alpha_i))$ is the i^{th} coordinate point of \mathbb{P}^{d-1} . The final part follows from the fact that $D_{\mathcal{A}}|_{E_p}$ is a divisor on E_p of degree $D_{\mathcal{A}} \cdot E_p = \mu(p)$, and $E_p \simeq \mathbb{P}^1$. \square

3.1. Castelnuovo–Mumford regularity and graded betti numbers. The Castelnuovo–Mumford regularity of a coherent sheaf \mathcal{F} on \mathbb{P}^n is usually phrased in terms of vanishing of certain cohomology modules. Letting $N = \oplus_n H^0(\mathcal{F}(n))$, we may [8] rephrase the condition as

Definition 3.5. *For a polynomial ring R , a finitely generated, graded R -module N has Castelnuovo–Mumford regularity j if j is the smallest number such that $\text{Tor}_i^R(N, \mathbb{C})_{i+j+1} = 0$ for all i . The graded betti numbers of a graded R -module N are indexed by*

$$b_{ij} = \dim_{\mathbb{C}} \text{Tor}_i^R(N, \mathbb{C})_j.$$

Example 3.6. We revisit Example 2.1. The four triple points yield four quadratic generators for the Orlik-Terao ideal I . These four quadrics generate I (see [34]), and a computation in `Macaulay2` yields the graded betti numbers of $C(\mathcal{A})$:

total	1	4	5	2
0	1	–	–	–
1	–	4	2	–
2	–	–	3	2

This diagram is read as follows: the entry in position (i, j) is simply $b_{i,i+j}$, e.g.

$$\dim_{\mathbb{K}} \operatorname{Tor}_2^R(C(\mathcal{A}), \mathbb{C})_4 = 3.$$

The betti table has a very nice interpretation in terms of Castelnuovo-Mumford regularity: the regularity is the index of the last nonzero row.

Theorem 3.7. *For $\mathcal{A} \subseteq \mathbb{P}^n$, $C(\mathcal{A})$ is n -regular.*

Proof. In [27], Proudfoot and Speyer show that the Orlik-Terao algebra is Cohen-Macaulay (for $n = 2$ this also follows from Theorem 3.3). Thus, there exists a regular sequence on $C(\mathcal{A})$ of $\dim(V(I)) + 1 = n + 1$ linear forms; quotienting by this sequence yields an Artinian ring whose Hilbert series is the numerator of the Hilbert series of $C(\mathcal{A})$. The regularity of an Artinian module is equal to the length of the module, so the result follows from Equation 5. \square

It follows easily from Theorem 3.7 and Terao's work in [38] that

Proposition 3.8. *For $\mathcal{A} \subseteq \mathbb{P}^n$ with $|\mathcal{A}| = d$, if $I = I_2$, then*

$$\dim_{\mathbb{C}} \operatorname{Tor}_2^R(C(\mathcal{A}), \mathbb{C})_3 = 2 \left(\binom{d}{3} - 1 \right) - (d - 3) \left(\sum_{p_i \in L_2(\mathcal{A})} \mu(p_i) + 1 \right).$$

Example 3.9. The A_3 arrangement is supersolvable, so by [34] $I = I_2$, and Proposition 3.8 shows there are two linear first syzygies on I . This explains the top row of the betti table in Example 3.6.

4. NETS, SYZYGIES, AND SCROLLS

In [21], Libgober-Yuzvinsky found a surprising connection between nets and the first resonance variety. The approach was further developed by Yuzvinsky in [42], with a beautiful complete picture emerging in Falk and Yuzvinsky's paper on multineets [12]. In this section, we connect nets to the linear syzygies of $C(\mathcal{A})$, and hence to $R^1(\mathcal{A})$. This allows us to give an interpretation of the first resonance variety in terms of the geometry of $X_{\mathcal{A}}$.

Suppose Z is a subset of the intersection points of \mathcal{A} , and let J denote the $|Z| \times d$ incidence matrix of points and lines and E denote a $d \times d$ matrix with every entry one. If \widehat{Z} is the blowup of \mathbb{P}^2 at the points of Z , then [21] shows that

$$J^t J - E = Q(\widehat{Z})$$

is the intersection form on \widehat{Z} , and is a generalized Cartan matrix. Using the Vinberg classification of such matrices [19], they show that any component of $R^1(\mathcal{A})$ corresponds to a choice of points Z such that $Q(\widehat{Z})$ consists of at least three affine blocks, with no finite or indefinite blocks, and the block sum decomposition of $Q(\widehat{Z})$ yields a neighborly partition. Before going into the details of the connection between multineets, divisors and syzygies, we give a pair of motivating examples.

Example 4.1. The matroid $(9_3)_2$ of Hilbert and Cohn-Vossen is realized below by $\mathcal{A} = V(xyz(x+y)(y+z)(x+3z)(x+2y+z)(x+2y+3z)(2x+3y+3z))$. It has nine triple points and nine double points, thus $P(\mathcal{A}, t) = (1+t)(1+8t+19t^2)$.

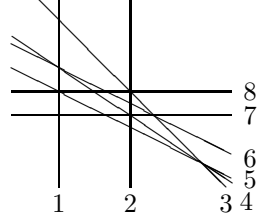


FIGURE 2. An arrangement realizing $(9_3)_2$

The graded betti numbers for $C((9_3)_2)$ are:

total	1	11	75	156	145	66	12
0	1	—	—	—	—	—	—
1	—	9	—	—	—	—	—
2	—	2	75	156	145	66	12

Example 4.2. The $(9_3)_1$ matroid of Hilbert and Cohn-Vossen is realized below by $\mathcal{A} = V(xyz(x-y)(y-z)(x-y-z)(2x+y+z)(2x+y-z)(2x-5y+z))$. It has nine triple points and nine double points, so $P((9_3)_1, t) = P((9_3)_2, t)$.

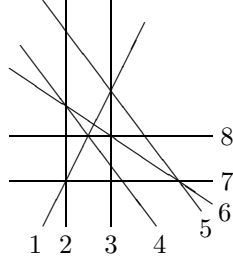


FIGURE 3. An arrangement realizing $(9_3)_1$

However, the graded betti numbers for $C((9_3)_1)$ are:

total	1	13	77	156	145	66	12
0	1	—	—	—	—	—	—
1	—	9	2	—	—	—	—
2	—	4	75	156	145	66	12

The arrangement $(9_3)_1$ possesses a pair of linear first syzygies, while $(9_3)_2$ has no linear first syzygies. An easy check shows that $(9_3)_1$ admits a neighborly partition $|169|258|347|$ and has a corresponding non-local component (see below) in $R^1(A)$, whereas $(9_3)_2$ does not. To better understand the connection between $R^1(A)$ and syzygies, we now review two constructions.

4.1. Nets and multinets. It is easy to see that any $p \in L_2(\mathcal{A})$ with $\mu(p) \geq 2$ yields a component $\mathbb{P}^{\mu(p)-1} \subseteq R^1(A)$. Such components are called *local components*. Components which are not of this type are called *essential*. In [42], Yuzvinsky used *nets* to analyze the essential components of $R^1(A)$.

Definition 4.3. Let $3 \leq k \in \mathbb{Z}$. A k -net in \mathbb{P}^2 is a pair (\mathcal{A}, Z) where \mathcal{A} is a finite set of distinct lines partitioned into k subsets $\mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i$ and Z is a finite set of points, such that:

- (1) for every $i \neq j$ and every $L \in \mathcal{A}_i$, $L' \in \mathcal{A}_j$, $L \cap L' \in Z$.
- (2) for every $p \in Z$ and every $i \in \{1, \dots, k\}$, \exists a unique $L \in \mathcal{A}_i$ containing p .

Thus, for a k -net, $|A_i| = |L \cap Z|$ for any block A_i and line $L \in \mathcal{A}$; denote this number by m . Following Yuzvinsky, we call m the order of the net, and refer to a k -net of order m as a (k, m) -net; note that $|Z| = m^2$. Yuzvinsky shows in [42] that a net must have $k \in \{3, 4, 5\}$, and improves this in [43] to $k \in \{3, 4\}$.

In [12], Falk and Yuzvinsky extend the notion of a net to a multinet; in a multinet lines may occur with multiplicity. Write \mathcal{A}_w for a multiarrangement, where $w \in \mathbb{N}^d$, and $w(L)$ denotes the multiplicity of a line.

Definition 4.4. A weak (k, m) -multinet on a multi-arrangement \mathcal{A}_w is a pair (Π, Z) where Π is a partition of \mathcal{A}_w into $k \geq 3$ classes A_1, \dots, A_k , and Z is a set of multiple points, such that

- (1) $\sum_{L \in A_i} w(L) = m$, independent of i .
- (2) For every $L \in A_i$ and $L' \in A_j$, with $i \neq j$, $L \cap L' \in Z$.
- (3) For each $p \in Z$, $\sum_{L \in A_i, p \in L} w(L)$ is a constant n_p , independent of i .

A multinet is a weak multinet satisfying the additional property

- (4) For $i \in \{1, \dots, k\}$ and $L, L' \in A_i$, \exists a sequence $L = L_0, L_1, \dots, L_r = L'$ such that $L_{j-1} \cap L_j \notin Z$ for $1 \leq j \leq r$.

Example 4.5. The reflection arrangement of type B_3 is depicted below (there is also a line at infinity). Falk and Yuzvinsky show that this arrangement supports a multinet which is not a net: assign weight two to lines $(3, 6, 8)$ and weight one to the remaining lines.

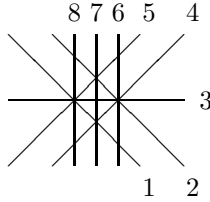


FIGURE 4. The B_3 -arrangement

The following lemma of [12] will be useful:

Lemma 4.6. Suppose (\mathcal{A}_w, Z) is a weak (k, m) -multinet. Then

- (1) $\sum_{L \in \mathcal{A}_w} w(L) = km$.
- (2) $\sum_{p \in Z} n_p^2 = m^2$
- (3) For each $L \in \mathcal{A}_w$, $\sum_{p \in Z \cap L} n_p = m$.

4.2. Determinantal syzygies and factoring divisors. One simple way in which linear syzygies can arise comes from a factorization of divisors. First, a definition

Definition 4.7. *A matrix of linear forms is 1 – generic if it has no zero entry, and cannot be transformed by row and column operations to have a zero entry.*

For $Y \subseteq \mathbb{P}^n$ irreducible and linearly normal, if there exist line bundles \mathcal{L}_1 and \mathcal{L}_2 such that $\mathcal{O}_Y(1) = \mathcal{L}_1 \otimes \mathcal{L}_2$ with $h^0(\mathcal{L}_i) = a_i$, then the $a_1 \times a_2$ matrix γ representing the multiplication table

$$H^0(\mathcal{L}_1) \otimes H^0(\mathcal{L}_2) \longrightarrow H^0(\mathcal{O}_Y(1))$$

is 1–generic. More explicitly (see [8]), if

$$H^0(\mathcal{L}_1) = \text{Span}_{\mathbb{C}}\{e_1, \dots, e_{a_1}\} \text{ and } H^0(\mathcal{L}_2) = \text{Span}_{\mathbb{C}}\{f_1, \dots, f_{a_2}\},$$

then γ has (i, j) entry $e_i \otimes f_j$, corresponding to a linear form on \mathbb{P}^n , and elements of the ideal $I_2(\gamma)$ of 2×2 minors of γ vanish on Y . The most familiar example occurs when $a_1 = 2$ and $a_2 = k$. In this case, the minimal free resolution of $I_2(\gamma)$ is an Eagon-Northcott complex. This relates to geometry via scrolls: let Ψ be the locus of points where a 1–generic matrix

$$\gamma = \begin{bmatrix} l_1 & \cdots & l_k \\ m_1 & \cdots & m_k \end{bmatrix}$$

has rank one. If

$$L_{[\lambda:\nu]} = \{p \in \mathbb{P}^n \mid \lambda l_1(p) + \mu m_1(p) = \cdots = \lambda l_k(p) + \mu m_k(p) = 0\},$$

then (see 9.10 of [16])

$$\Psi = \bigcup_{[\lambda:\nu] \in \mathbb{P}^1} L_{[\lambda:\nu]},$$

where $L_{[\lambda:\nu]} \simeq \mathbb{P}^{n-k}$, so Ψ is a union of linear spaces. Geometrically, the zero locus of the 2×2 minors of γ is a scroll which contains $V(I_Y)$.

4.3. Connecting nets and determinantal syzygies. The computation in the proof of Theorem 3.3 and the fact that $h^1(D) \geq 0$ shows that if $D_{\mathcal{A}} = A + B$ with $A = mE_0 - \sum a_i E_i$, then

$$h^0(A) \geq \binom{m+2}{2} - \sum_{p \in L_2(\mathcal{A})} \binom{a_i+1}{2}, \quad h^0(B) \geq \binom{d+1-m}{2} - \sum_{p \in L_2(\mathcal{A})} \binom{\mu(p)-a_i+1}{2}.$$

For an arrangement \mathcal{A} , if there exists a choice of parameters m and a_i such that $h^0(A) = a \geq 2$ and $h^0(B) = b \geq 3$, then the results of the previous section show that there will exist linear first syzygies on I .

Example 4.8. We revisit Example 4.2. Let $A = 3E_0 - \sum_{\{p \mid \mu(p)=2\}} E_p$. Clearly $A^2 = AK = 0$, so we can only guarantee that $h^0(A) \geq 1$. In fact, $h^0(A) = 2$, hence $h^1(A) = 1$. To see this, note that a direct computation shows that the space of cubics passing through the nine multiple points of \mathcal{A} is two dimensional. Since

$$\text{Span}_{\mathbb{C}}(L_1 L_6 L_9, L_3 L_4 L_7, L_2 L_5 L_8) \subseteq H^0(A),$$

and any two of these are independent, we see that the sections are given by the net. Next, consider the residual divisor $B = D_{\mathcal{A}} - A$. Since $B^2 = 16 - 18 = -2$ and $-BK = 15 - 9 = 6$, we have that $h^0(B) \geq 3$. In fact, equality holds, so I contains the 2×2 minors of a 2×3 matrix of linear forms, explaining the linear syzygies.

Lemma 4.9. *A (k, m) multinet gives a divisor A on X such that $h^0(A) = 2$.*

Proof. Let

$$A = mE_0 - \sum_{p \in Z} n_p E_p.$$

Condition (1) of Definition 4.4 implies that for each block A_i of the multinet, $\prod_{L \in A_i} L^{w(L)}$ is homogeneous of degree m , and Condition (3) shows that it vanishes to order exactly n_p on E_p . In particular, this shows that $\prod_{L \in A_i} L^{w(L)} \in H^0(A)$. If all the blocks of the partition were independent, then this would imply that $h^0(A) \geq k$, but it turns out that the sections are all fibers of a pencil of plane curves, which follows from Theorem 3.11 of [12]. \square

In [12], Falk and Yuzvinsky show that the following are equivalent:

- (1) $R^1(A)$ contains a nonlocal component $\simeq \mathbb{P}^{k-2}$.
- (2) \mathcal{A} supports a (k, m) multinet.
- (3) \exists a pencil of plane curves with connected fibers, with at least three fibers (loci of) products of linear forms, and \mathcal{A} is the union of all such fibers.

In general, determining the dimension of $h^1(D)$ for $D \in \text{Pic}(X)$ is not easy. However, in the special case of a net, there is enough information to give a lower bound for the dimension of the sections of the residual divisor which is often exact.

Theorem 4.10. *If \mathcal{A} is a (k, m) net, then $D_{\mathcal{A}} = A + B$, with*

$$h^0(A) = 2 \text{ and } h^0(B) \geq km - \binom{m+1}{2}.$$

Proof. For a (k, m) net, all lines occur with multiplicity one. Let

$$A = mE_0 - \sum_{p \in Z} E_p.$$

By Lemma 4.9, $h^0(A) = 2$. Since $B = D_{\mathcal{A}} - mE_0 + \sum_{p \in Z} E_p$, $\frac{B^2 - BK}{2} + 1$

$$\begin{aligned} &= \frac{(d-m-1)(d-m-1+3)+2}{2} + \left(\sum_{p \in L_2(\mathcal{A})} \mu(p) E_p + \sum_{p \in Z} E_p \right) \left(\sum_{p \in L_2(\mathcal{A})} (\mu(p) - 1) E_p + \sum_{p \in Z} E_p \right) \\ &= \binom{d+1-m}{2} - \binom{d}{2} + \sum_{p \in Z} \mu(p) \\ &= \binom{m}{2} - d(m-1) + \sum_{p \in Z} \mu(p). \end{aligned}$$

We now compute that

$$\begin{aligned} \sum_{p \in Z} \mu(p) + |Z| &= \sum_{p \in Z} (\mu(p) + 1) \\ &= k \sum_{p \in Z} n_p \\ &= km^2. \end{aligned}$$

The second line follows since n_p lines from each block A_i pass through p , and there are k blocks. The third line follows from Lemma 4.6 and the fact that $n_p = 1$ for a net, hence $n_p^2 = n_p$. Since for a (k, m) -net $|Z| = m^2$,

$$\sum_{p \in Z} \mu(p) = (k-1)m^2 = dm - m^2.$$

Combining this with the previous calculation shows that for a (k, m) -net

$$h^0(B) = h^1(B) + \binom{m}{2} - d(m-1) + dm - m^2 \geq d - \binom{m+1}{2}.$$

Recalling that $km = d$ concludes the proof. \square

Corollary 4.11. *If \mathcal{A} is a (k, m) net with $k \geq m$, then I contains the 2×2 minors of a 1-generic $2 \times \left(km - \binom{m+1}{2}\right)$ matrix. Thus the resolution of I contains an Eagon-Northcott complex as a subcomplex.*

Proof. Since $k \in \{3, 4\}$, if $(k, m) = (3, 2)$ or $(3, 3)$ then by Theorem 4.10 $h^0(B) \geq 3$, and if $(k, m) = (4, 3)$ or $(4, 4)$ then $h^0(B) \geq 6$. Note that the only known example of a 4-net is the $(4, 3)$ net corresponding to the Hessian configuration. \square

Example 4.12. For the arrangement A_3 appearing in Example 2.1, Z is the collection of multiple points, and

$$A = 2E_0 - \sum_{\{p \mid \mu(p)=2\}} E_p$$

and

$$B = 3E_0 - \sum_{p \in L_2(\mathcal{A})} E_p.$$

So $d - \binom{m+1}{2} = 6 - 3 = 3$ and I contains the 2×2 minors of a 2×3 matrix.

5. CONNECTION TO DERIVATIONS

In this section, we show that the generators of the Jacobian ideal of $\mathcal{A} \subseteq \mathbb{P}^2$ are contained in $H^0(D_{\mathcal{A}})$, and that the associated projection map $X \rightarrow \mathbb{P}^2$ has degree $\sum_{p \in L_2(\mathcal{A})} \mu(p) - |\mathcal{A}| + 1$. This relates X to one of the fundamental objects in arrangement theory: the module $D(\mathcal{A})$ of derivations tangent to \mathcal{A} .

Definition 5.1. $D(\mathcal{A}) = \{\theta \mid \theta(\alpha_i) \in \langle \alpha_i \rangle \text{ for all } \alpha_i \text{ such that } V(\alpha_i) \in \mathcal{A}\}.$

The module $D(\mathcal{A})$ is a graded $S = \mathbb{C}[x_0, \dots, x_n]$ module, and over a field of characteristic zero, $D(\mathcal{A}) \simeq E \oplus D_0(\mathcal{A})$, where E is the Euler derivation and $D_0(\mathcal{A})$ corresponds to the module of syzygies on the Jacobian ideal J_{α} of the defining polynomial $\alpha = \prod_{i=1}^d \alpha_i$ of \mathcal{A} . An arrangement \mathcal{A} is *free* if $D(\mathcal{A}) \simeq \oplus S(-a_i)$; the a_i are called the *exponents* of \mathcal{A} . Terao's theorem [37] is that if $D(\mathcal{A}) \simeq \oplus S(-a_i)$, then $P(M, t) = \prod (1 + a_i t)$. Supersolvable arrangements are free, so Example 2.1 is of this type, and

$$P(A_3, t) = (1 + t)(1 + 2t)(1 + 3t)$$

On the other hand, the arrangement \mathcal{A} of Example 2.5 is not free, and $P(\mathcal{A}, t)$ does not factor. However, it is shown in [30] that for $\mathcal{A} \subseteq \mathbb{P}^2$ the Poincaré polynomial is $(1 + t) \cdot c_t(D_0^{\vee})$, where c_t is the Chern polynomial and D_0^{\vee} is the dual of the rank two vector bundle associated to D_0 . An easy localization argument [31] shows that in this case, the Jacobian ideal is a local complete intersection with

$$\deg(J_{\alpha}) = \sum_{p \in L_2(\mathcal{A})} \mu(p)^2.$$

For $\mathcal{A} \subseteq \mathbb{P}^n$ with $n \geq 3$, some generalizations are possible, see [22].

Proposition 5.2. *For an arrangement $\mathcal{A} \subseteq \mathbb{P}^2$,*

$$J_\alpha \subseteq L = \langle l_1, \dots, l_d \rangle = \left\langle \frac{\alpha}{\alpha_1}, \dots, \frac{\alpha}{\alpha_d} \right\rangle.$$

Proof. By Lemma 3.2,

$$L = \bigcap_{p \in L_2(\mathcal{A})} I_p^{\mu(p)} \subseteq \mathbb{C}[x, y, z].$$

The ideal J_α is generated in degree $d-1$. The result of [31] mentioned above implies that at any point $p \in L_2(\mathcal{A})$, the localization $(J_\alpha)_p$ is a local complete intersection: changing coordinates so that $p = (0 : 0 : 1)$, and writing $\mathcal{A} = V(L_0 L_1)$ with L_0 the product of the defining linear forms which vanish at p and L_1 the product of the remaining forms, we have

$$(J_\alpha)_p = \langle \partial(L_0)/\partial x, \partial(L_0)/\partial y \rangle.$$

In particular, both generators are of degree $\mu(p)$, so in the primary decomposition of J_α , the primary component associated to $I(p)$ is contained in $I_p^{\mu(p)}$. Now,

$$\text{Sing}(\mathcal{A}) = \bigcup_{I(p) \in \text{Ass}(\sqrt{J_\alpha})} V(I(p)).$$

If Q_p is the $I(p)$ -primary component of J_α , then

$$\bigcap_{I(p) \in S} Q_p \subseteq L.$$

Since the saturation of J_α with respect to $\langle x, y, z \rangle$ is the left hand side, and J_α is generated in degree $d-1$, the result follows. \square

The inclusion $W = J_\alpha \subseteq H^0(D_{\mathcal{A}})$ corresponds to an induced map

$$\begin{array}{ccc} \mathbb{P}^2 \setminus V(\alpha) & \xrightarrow{\phi_{\mathcal{A}}} & \mathbb{P}(H^0(D_{\mathcal{A}})) \\ & \searrow \psi & \downarrow \pi \\ & & \mathbb{P}(W) \end{array}$$

Proposition 5.3. *The degree of π is $\sum_{p \in L_2(\mathcal{A})} \mu(p) - |\mathcal{A}| + 1$.*

Proof. In [7], Dimca and Papadima show that on a projective hyperplane complement $\mathbb{P}^n \setminus V(\alpha)$, the degree of the gradient map ψ

$$\mathbb{P}^n \setminus V(\alpha) \xrightarrow{\left[\begin{array}{ccc} \partial(\alpha)/\partial x_0 & \cdots & \partial(\alpha)/\partial x_n \end{array} \right]} \mathbb{P}^n$$

is equal to $b_n(\mathbb{P}^n \setminus V(\alpha))$; for a configuration of $\mathcal{A} \subseteq \mathbb{P}^2$ this means the degree of the gradient map is

$$\sum_{p \in L_2(\mathcal{A})} \mu(p) - |\mathcal{A}| + 1.$$

By Theorem 3.4, $\phi_{\mathcal{A}}$ is an isomorphism on $\mathbb{P}^2 \setminus V(\alpha)$, and the result follows. \square

Concluding Remarks and Questions

- (1) To study $Tor_i^R(C(\mathcal{A}), \mathbb{C})_{i+1}$, it suffices to restrict to the case of line arrangements. This follows since the quadratic generators of $C(\mathcal{A})$ depend only on $L_2(\mathcal{A})$, hence taking the intersection of \mathcal{A} with a generic \mathbb{P}^2 leaves these generators (and relations among them) unchanged. The graded betti numbers for I_2 are not combinatorial invariants; it would be interesting to understand how the geometry of \mathcal{A} governs b_{ij} , even for $j = i + 1$.
- (2) Can freeness of $D(\mathcal{A})$ be related to the surface X and divisor $D_{\mathcal{A}}$? It seems possible that there is a connection between $D_{\mathcal{A}}$ and multiarrangements, studied recently in [1], [39], [40].
- (3) For $n \geq 3$, is $C(\mathcal{A})$ the homogeneous coordinate ring of a blowup of \mathbb{P}^n along a locus related to the arrangement \mathcal{A} ? Since $C(\mathcal{A})$ is Cohen-Macaulay, if this is true, then combining Riemann-Roch with Terao's result would yield a formula for the global sections of $D_{\mathcal{A}}$.

Acknowledgements Computations were performed using Macaulay2, by Grayson and Stillman, and available at: <http://www.math.uiuc.edu/Macaulay2/>. Many of the examples in this paper appear in Suciu's survey [36]; that paper is also an excellent reference for the many questions on resonance varieties not treated here. This work began during a visit to the Fields institute for a conference in honour of Peter Orlik.

REFERENCES

- [1] T. Abe, H. Terao, M. Wakefield, *The e -multiplicity and addition-deletion theorems for multiarrangements*, J. Lond. Math. Soc. **77** (2008), 335–348.
- [2] W. Arvola, *The fundamental group of the complement of an arrangement of complex hyperplanes*, Topology **31** (1992), 757–765.
- [3] D. Cohen, A. Suciu, *The braid monodromy of plane algebraic curves and hyperplane arrangements*, Comment. Math. Helv. **72** (1997), 285–315.
- [4] D. Cohen, A. Suciu, *Characteristic varieties of arrangements*, Math. Proc. Cambridge Phil. Soc. **127** (1999), 33–53.
- [5] E. Davis, A. Geramita, *Birational morphisms to \mathbb{P}^2 : an ideal-theoretic perspective*, Math. Ann. **279** (1988), 435–448.
- [6] C. De Concini, C. Procesi, *Wonderful models of subspace arrangements*. Selecta Math. **1** (1995), no. 3, 459–494.
- [7] A. Dimca, S. Papadima, *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements*, Ann. Math. **158** (2003), 473–507.
- [8] D. Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, Berlin-Heidelberg-New York, 2005.
- [9] D. Eisenbud, *Linear sections of determinantal varieties*, Amer. J. Math. **110** (1988), no. 3, 541–575.
- [10] M. Falk, *Arrangements and cohomology*, Ann. Combin. **1** (1997), 135–157.
- [11] M. Falk, *Resonance varieties over fields of positive characteristic*, IMRN **3** (2007)
- [12] M. Falk, S. Yuzvinsky, *Multinets, resonance varieties, and pencils of plane curves*, Compos. Math. **143** (2007), 1069–1088 MR 2339840
- [13] H. Hamm, D. T. Lê, *Un théorème de Zariski du type de Lefschetz*, Ann. Sci. École Norm. Sup. **6** (1973), 317–366.
- [14] B. Harbourne, *Problems and progress: survey on fat points in \mathbb{P}^2* , Queens Papers in Pure and Applied Mathematics **123** (2002), 85–132.
- [15] B. Harbourne, *Free resolutions of fat point ideals on \mathbb{P}^2* . J. Pure Appl. Algebra **125** (1998), 213–234.
- [16] J. Harris, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, Berlin-Heidelberg-New York, 1992.
- [17] F. Hirzebruch, *Arrangements of lines and algebraic surfaces*, In: Arithmetic and Geometry, vol. II, Progress in Math., vol. 36, Birkhäuser, Boston, 1983, pp. 113–140.

- [18] P. Lima-Filho, H. Schenck, *The holonomy Lie algebra of subarrangements of A_n* , IMRN, (2009), 1421-1432.
- [19] V. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 1990.
- [20] J. Koh, M. Stillman, *Linear syzygies and line bundles on an algebraic curve*, J. Algebra **125** (1989), 120-132.
- [21] A. Libgober, S. Yuzvinsky, *Cohomology of the Orlik-Solomon algebras and local systems*, Compositio Math. **121** (2000), 337-361.
- [22] M. Mustață, H. Schenck, *The module of logarithmic p -forms of a locally free arrangement*, J. Algebra, **241** (2001), 699-719.
- [23] P. Orlik, L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), 167-189.
- [24] P. Orlik, H. Terao, *Arrangements of Hyperplanes*, Grundlehren Math. Wiss., Bd. 300, Springer-Verlag, Berlin-Heidelberg-New York, 1992.
- [25] P. Orlik, H. Terao, *Commutative algebras for arrangements*, Nagoya Math Journal, **134** (1994), 65-73.
- [26] S. Papadima, A. Suciu, *Chen Lie algebras*, IMRN **2004:21** (2004), 1057-1086.
- [27] N. Proudfoot, D. Speyer, *A broken circuit ring*, Beiträge zur Algebra und Geometrie, **47** (2006), 161-166.
- [28] R. Randell, *The fundamental group of the complement of a union of complex hyperplanes*, Invent. Math. **69** (1982), 103-108.
- [29] M. Salvetti, *Topology of the complement of real hyperplanes in \mathbb{C}^N* , Invent. Math. **88** (1987), 603-618.
- [30] H. Schenck, *A rank two vector bundle associated to a three arrangement, and its Chern polynomial*, Advances in Mathematics, **149** (2000), 214-229.
- [31] H. Schenck, *Elementary modifications and line configurations in \mathbb{P}^2* , Commentarii Mathematici Helvetici, **78** (2003), 447-462.
- [32] H. Schenck, A. Suciu, *Lower central series and free resolutions of hyperplane arrangements*, Trans. Amer. Math. Soc. **354** (2002), 3409-3433.
- [33] H. Schenck, A. Suciu, *Resonance, linear syzygies, Chen groups, and the Bernstein-Gelfand-Gelfand correspondence*, Trans. Amer. Math. Soc. **358** (2006), 2269-2289.
- [34] H. Schenck, S. Tohaneanu, *The Orlik-Terao algebra and 2-formality*, Math. Res. Lett. **16** (2009), 171-182.
- [35] R. Stanley, *Supersolvable lattices*, Algebra Universalis **2** (1972), 197-217.
- [36] A. Suciu, *Fundamental groups of line arrangements: Enumerative aspects*, Contemporary Math. **276** (2001), AMS, Providence, RI, 43-79.
- [37] H. Terao, *Generalized exponents of a free arrangement of hyperplanes and Shepard-Todd-Brieskorn formula*, Inventiones Mathematicae **63** (1981), 159-179.
- [38] H. Terao, *Algebras generated by reciprocals of linear forms*, J. Algebra, **250** (2002), 549-558.
- [39] M. Yoshinaga, *Characterization of a free arrangement and conjecture of Edelman and Reiner*, Invent. Math. **157** (2004), no. 2, 449-454.
- [40] M. Yoshinaga, *On the freeness of 3-arrangements*, Bull. London Math. Soc. **37** (2005), no. 1, 126-134.
- [41] S. Yuzvinsky, *First two obstructions to the freeness of arrangements*, Transactions of the A.M.S., **335** (1993), 231-244.
- [42] S. Yuzvinsky, *Realization of finite Abelian groups by nets in \mathbb{P}^2* , Compos. Math. **140** (2004), 1614-1624.
- [43] S. Yuzvinsky, *A new bound on the number of special fibers in a pencil of plane curves*, Proc. Amer. Math. Soc. **137** (2009), 1641-1648.

SCHENCK: MATHEMATICS DEPARTMENT, UNIVERSITY OF ILLINOIS, URBANA, IL 61801, USA
 E-mail address: `schenck@math.uiuc.edu`